

## Fermion Bags in the Massive Gross-Neveu Model

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### Abstract

As has long been known, it is energetically favorable for massive fermions to deform the homogeneous vacuum around them, giving rise to extended bag-like objects. We study this phenomenon non-perturbatively in a model field theory, the 1+1 dimensional Massive Gross-Neveu model, in the large  $N$  limit. We prove that the bags in this model are necessarily time dependent. We calculate their masses variationally and demonstrate their stability. We find a non-analytic behavior in these masses as we approach the standard massless Gross-Neveu model and argue that this behavior is caused by the kink-antikink threshold. This work extends our previous work to a non-integrable field theory.

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A central concept in particle physics states that fundamental particles acquire their masses through interactions with vacuum condensates. Thus, a massive particle may carve out around itself a spherical region [1] or a shell [2] in which the condensate is suppressed, thus reducing the effective mass of the particle at the expense of volume

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and gradient energy associated with the condensate. This picture has interesting phenomenological consequences [1, 3].

Here we study these effects within the 1 + 1 dimensional massive generalization of the Gross-Neveu model [4] (which we will refer to as MGN),

$$\begin{aligned} S &= \int d^2x \left\{ \sum_{a=1}^N \bar{\psi}_a (i\not{\partial} - M) \psi_a + \frac{g^2}{2} \left( \sum_{a=1}^N \bar{\psi}_a \psi_a \right)^2 \right\} \\ &= \int d^2x \left\{ \bar{\psi} [i\not{\partial} - \sigma] \psi - \frac{1}{2g^2} (\sigma^2 - 2M\sigma) \right\} \end{aligned} \quad (1)$$

describing  $N$  self interacting massive Dirac fermions  $\psi_a$  carrying a flavor index  $a = 1, \dots, N$ , which we promptly suppress. As usual, the theory can be rewritten with the help of a scalar flavor singlet auxiliary field  $\sigma(x)$ . Also as usual, we take the large  $N$  limit holding  $\lambda \equiv Ng^2$  fixed. Integrating out the fermions, we obtain the bare effective action

$$S[\sigma] = -\frac{1}{2g^2} \int d^2x (\sigma^2 - 2M\sigma) - iN \text{Tr} \log (i\not{\partial} - \sigma). \quad (2)$$

Noting that  $\gamma_5(i\not{\partial} - \sigma) = -(i\not{\partial} + \sigma)\gamma_5$ , we can rewrite the  $\text{Tr} \log(i\not{\partial} - \sigma)$  as  $\frac{1}{2} \text{Tr} \log(i\not{\partial} - \sigma)(i\not{\partial} + \sigma)$ . If  $\sigma$  is time independent, this may be further simplified to  $\frac{1}{2} \int \frac{d\omega}{2\pi} [(\text{Tr} \log(h_+ - \omega^2) + \text{Tr} \log(h_- - \omega^2))]$  where  $h_{\pm} \equiv -\partial_x^2 + \sigma^2 \pm \sigma'$ . Clearly, the two Schrödinger operators  $h_{\pm}$  are isospectral (see Sec. II of [6]) and thus we obtain

$$S[\sigma] = -\frac{1}{2g^2} \int d^2x (\sigma^2 - 2M\sigma) - iN \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \log(h_- - \omega^2) \quad (3)$$

In contrast to the standard massless Gross-Neveu model (the GN model), the MGN model studied here is not invariant under the  $Z_2$  symmetry  $\psi \rightarrow \gamma_5 \psi$ ,  $\sigma \rightarrow -\sigma$ , and the physics is correspondingly quite different. The GN model contains a soliton (the so called CCGZ kink [7, 8, 6]) in which the  $\sigma$  field takes on equal and opposite values at  $x = \pm\infty$ . The stability of this soliton is obviously guaranteed by topological considerations. With any non-zero  $M$  the vacuum value of  $\sigma$  is unique and the CCGZ kink becomes infinitely massive and disappears. If any soliton exists at all, its stability has to depend on the energetics of trapping fermions. Also, the GN model is

completely integrable, while the MGN model is widely believed to be non-integrable. In recent work [5, 6] we have studied integrable models, and one of the purposes of this note is to show that it is possible to obtain non-perturbative results even for non-integrable models, albeit in the large  $N$  limit.

*The vacuum state* Setting  $\sigma$  to a constant we obtain from (3) the renormalized effective potential (per flavor)

$$V(\sigma, \mu) = \frac{\sigma^2}{4\pi} \log \frac{\sigma^2}{e\mu^2} + \frac{1}{\lambda(\mu)} \left[ \frac{\sigma^2}{2} - M(\mu)\sigma \right], \quad (4)$$

where  $\mu$  is a sliding renormalization scale with  $\lambda(\mu) = Ng^2(\mu)$  and  $M(\mu)$  the running couplings. By equating the coefficient of  $\sigma^2$  in two versions of  $V$ , one defined with  $\mu_1$  and the other with  $\mu_2$ , we find immediately that

$$\frac{1}{\lambda(\mu_1)} - \frac{1}{\lambda(\mu_2)} = \frac{1}{\pi} \log \frac{\mu_1}{\mu_2} \quad (5)$$

and thus the coupling  $\lambda$  is asymptotically free, just as in the GN model. Furthermore, by equating the coefficient of  $\sigma$  in  $V$  we see that the ratio  $\frac{M(\mu)}{\lambda(\mu)}$  is a renormalization group invariant. Thus,  $M$  and  $\lambda$  have the same scale dependence.

Without loss of generality we assume that  $M(\mu) > 0$  and thus the absolute minimum of (4), namely, the condensate  $m = \langle \sigma \rangle$ , is the positive solution of the gap equation

$$\frac{dV}{d\sigma} \Big|_{\sigma=m} = m \left[ \frac{1}{\pi} \log \frac{m}{\mu} + \frac{1}{\lambda(\mu)} \right] - \frac{M(\mu)}{\lambda(\mu)} = 0. \quad (6)$$

Referring to (1), we see that  $m$  is the mass of the fermion. Using (5), we can rewrite the gap equation as  $\frac{m}{\lambda(m)} = \frac{M(\mu)}{\lambda(\mu)}$ , which shows manifestly that  $m$ , an observable physical quantity, is a renormalization group invariant. This equation also implies that  $M(m) = m$ , which makes sense physically.

*Static space dependent  $\sigma(x)$  backgrounds* Ideally, we would like to solve the field equation  $\frac{\delta S}{\delta \sigma(x,t)} = 0$ , a difficult task beyond the capability of field theorists at present. A more realistic goal is to restrict ourselves to time-independent  $\sigma$  field and to try to solve  $\frac{\delta S}{\delta \sigma(x)} = 0$ , but even that is difficult since we don't know how to evaluate  $S$  for

an arbitrary time independent but space dependent  $\sigma(x)$ . Furthermore, we can show (see below) that such a solution does not exist.

The relevant physics is not difficult to understand. A generic  $\sigma(x)$  will distort the fermion vacuum, causing the fermions to back-react on  $\sigma(x)$  so as to minimize their energy, and in general  $\sigma$  will become time dependent. In our previous work [5] we have found a necessary condition (albeit generically insufficient) to avoid such a back-reaction in 1+1 dimensional theories. The condition is physical and easy to state (and we will state it in the present context.) Consider the expectation value of the fermionic vector current  $j^\mu(x)$  in the background specified by a field configuration  $\sigma$ . After some standard manipulations we could show [5] that the spatial component of the fermion number current will not vanish at spatial infinity, unless the Schrödinger operator  $h_-$  is reflectionless. Moreover, the fermion number current will run in opposite directions at  $x = \pm\infty$ . This apparent current non-conservation indicates that a state giving rise to a static reflectionful  $h_-$  is highly unstable and will immediately try to decay to a stable state by emitting fermions.

We will thus restrict ourselves to only those  $\sigma$  configurations which correspond to reflectionless  $h_-$ . Since there are only denumerably infinite number of reflectionless Schrödinger operators known, this condition vastly restricts the space of possible  $\sigma(x)$ . Our calculation amounts to a variational calculation in quantum field theory. For any given fermion number  $N_f$  the energy of the bag or lump we calculate below is an upper bound to the true energy.

*A variational calculation of the bag mass* This upper bound on the true energy cannot be saturated by static  $\sigma(x)$  configurations, because as we already mentioned, the MGN model does not have static saddle point  $\sigma(x)$  configurations. However, it is clear from the discussion above that reflectionless  $\sigma(x)$  configurations are the best trial configuration among all static configurations. As usual, the art behind a variational calculation consists of a judicious choice of a trial function.

The energy functional (per flavor)  $\mathcal{E}[\sigma(x)]$  for static  $\sigma(x)$  configurations is by

definition  $\mathcal{E} = -\frac{S}{NT}$  where  $T$  is some temporal infrared cutoff. We write (3) as [6]

$$\mathcal{E}[\sigma(x)] = \frac{1}{2\lambda} \int_{-\infty}^{\infty} dx [V(x) - 2M\sigma(x)] - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \text{Tr} \log [-\partial_x^2 + V(x) - \omega^2] \quad (7)$$

where  $V(x) = \sigma^2(x) - \sigma'(x)$ . (Here we used  $\int_{-\infty}^{\infty} dx \sigma'(x) = 0$  by invoking the boundary conditions  $\sigma(x) \xrightarrow{x \rightarrow \pm\infty} m$ .)

Out of the denumerably infinite number of reflectionless Schrödinger operators we now take the simplest possibility: that  $h_- = -\partial_x^2 + \sigma^2 - \sigma'$  has a single normalizable bound state at some positive [9] energy  $\omega_b^2 < m^2$  (and thus bound states at  $\pm\omega_b$  in the Dirac operator.) It is well-known from the annals of quantum mechanics that these properties uniquely determine the single parameter family of potentials

$$V(x) = m^2 - 2\kappa^2 \text{sech}^2 [\kappa(x - x_0)] \quad (8)$$

(up to an overall translation parameter  $x_0$  which we immediately set to zero.) The normalized bound state wave function is  $\psi_b(x) = \sqrt{\frac{\kappa}{2}} \text{sech } \kappa x$ . The bound state energy  $\omega_b^2$  is given by  $\kappa = \sqrt{m^2 - \omega_b^2}$ , thus suggesting that we trade  $\kappa$  immediately for an angle  $\frac{\pi}{2} \geq \theta \geq 0$  such that  $\kappa = m \sin \theta$  (and thus  $\omega_b = m \cos \theta$ .) The corresponding  $\sigma(x)$  is:

$$\sigma(x) = m + \kappa \left[ \tanh \left( \kappa x - \frac{1}{4} \log \frac{m + \kappa}{m - \kappa} \right) - \tanh \left( \kappa x + \frac{1}{4} \log \frac{m + \kappa}{m - \kappa} \right) \right]. \quad (9)$$

With (9) as a trial configuration, the energy (7) becomes an ordinary function  $\mathcal{E}(\theta)$ . We thus vary with respect to the variational parameter  $\theta$  (or equivalently  $\kappa$ .) The extremum condition on the energy is

$$\frac{\partial \mathcal{E}}{\partial \theta} = \int_{-\infty}^{\infty} dx \left\{ \left[ \frac{1}{2\lambda} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} R(x, \omega) \right] \frac{\partial V}{\partial \theta} - \frac{M}{\lambda} \frac{\partial \sigma}{\partial \theta} \right\} = 0. \quad (10)$$

Here  $R(x, \omega) \equiv \langle x | (-\partial_x^2 + \sigma^2 - \sigma' - \omega^2)^{-1} | x \rangle$  denotes the resolvent of  $h_-$ , and can be calculated to be [6]

$$R(x, \omega) = \frac{1}{2\sqrt{m^2 - \omega^2}} \left[ 1 + \frac{m^2 - \sigma^2 + \sigma'}{2(\omega_b^2 - \omega^2)} \right] = \frac{1}{2\sqrt{m^2 - \omega^2}} \left[ 1 + \frac{2\kappa\psi_b^2(x)}{\omega_b^2 - \omega^2} \right]. \quad (11)$$

Substituting (11) in (10) we find

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial \theta} &= \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2} \left[ \frac{1}{\lambda} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\sqrt{m^2 - \omega^2}} \right] \frac{\partial V}{\partial \theta} - \frac{M}{\lambda} \frac{\partial \sigma}{\partial \theta} \right\} \\ &- \kappa I(\omega_b, m) \langle \psi_b | \frac{\partial V}{\partial \theta} | \psi_b \rangle, \end{aligned} \quad (12)$$

where  $I(\omega_b, m) \equiv \int_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{1}{(\omega_b^2 - \omega^2) \sqrt{m^2 - \omega^2}}$  (the contour  $\mathcal{C}$  is specified below.)

Note that the  $\omega$  integral in the first term in (12) is logarithmically divergent. This UV divergence is taken care of as follows. Let us consider (12) at the cutoff scale  $\Lambda$ . Setting  $\sigma$  to its vacuum value in  $\delta \mathcal{E} / \delta \sigma(x) = 0$  (see (19) below) we have the (bare) gap equation

$$\frac{1 - \frac{M(\Lambda)}{m}}{\lambda(\Lambda)} = \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} \frac{1}{\sqrt{\omega^2 - m^2 + i\epsilon}}. \quad (13)$$

Using (13) in (12) we see that all reference to  $\Lambda$  disappears and the extremum condition becomes

$$\frac{\partial \mathcal{E}}{\partial \theta} = \frac{M}{2\lambda m} \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} dx [(\sigma - m)^2 - \sigma'] - \kappa I(\omega_b, m) \langle \psi_b | \frac{\partial V}{\partial \theta} | \psi_b \rangle. \quad (14)$$

To evaluate the integral  $I(\omega_b, m)$ , we have to choose the proper contour  $\mathcal{C}$ , and thus we have to invoke our understanding of the physics of fermions. We fill the Dirac sea, including the discrete state at  $-\omega_b$ , and then put  $N_f$  fermions into the state at  $\omega_b$ . Mathematically, we thus have to let  $\mathcal{C}$  enclose the cut on the negative  $\omega$  axis and then go around the pole at  $-\omega_b$   $N$  times and around the pole at  $\omega_b$   $N_f$  times. In this way, we obtain [6, 8]  $I = (\frac{2\theta}{\pi} - \nu)/m^2 \sin 2\theta$  where we have introduced the “filling fraction”  $\nu = \frac{N_f}{N}$ .

Recalling first order perturbation theory we immediately recognize the matrix-element in (14) as simply  $\partial \omega_b^2 / \partial \theta$ . Putting it all together we find the extremum condition

$$\frac{\partial \mathcal{E}}{\partial \theta} = 2m \left[ \left( \frac{\theta}{\pi} - \frac{\nu}{2} \right) + \gamma \tan \theta \right] \sin \theta = 0. \quad (15)$$

(where we have defined the renormalization group invariant ratio  $\gamma \equiv \frac{M}{\lambda m}$ ), thus fixing

$\theta$  as a function of the filling fraction

$$\frac{\theta}{\pi} + \gamma \tan\theta = \frac{\nu}{2}. \quad (16)$$

Integrating (14) and using (16) we find that the mass  $M$  (namely,  $N\mathcal{E}$  evaluated at the extremal point) of our bag or lump is

$$\frac{M(\nu, \gamma)}{Nm} = \frac{2}{\pi} \sin\theta + \gamma \log \frac{1 + \sin\theta}{1 - \sin\theta}. \quad (17)$$

By calculating  $\frac{d^2\mathcal{E}}{d\nu^2} = -\pi\sin\theta/(1 + \pi\gamma \sec^2\theta)$  we see that  $\mathcal{E}(\nu)$  is a convex function and thus satisfies  $\mathcal{E}(\nu_1 + \nu_2) < \mathcal{E}(\nu_1) + \mathcal{E}(\nu_2)$ . Therefore, a lump binding  $N\nu$  fermions is variationally stable against decaying into two lumps with  $N\nu_1$  and  $N\nu_2$  fermions respectively (with  $\nu = \nu_1 + \nu_2 < 1$ .) Thus, the lump binding  $N\nu$  fermions is the most variationally stable static configuration at the sector of fermion number  $N\nu$ . Note that this is true for small as well as for large values of  $\gamma$ . Furthermore, it is clear from (17) that the binding energy (in units of  $m$ ) per fermion  $B(\nu, \gamma) = 1 - \frac{M(\nu, \gamma)}{m\nu}$  increases with  $\nu$ , and does not saturate as in nuclear physics. This is characteristic of the bag picture, in which each additional particle digs a deeper hole in the vacuum condensate (“the mattress effect”). To demonstrate these facts we present in Fig. (1) a numerical computation of the binding energy per fermion at a particular  $\gamma$ .

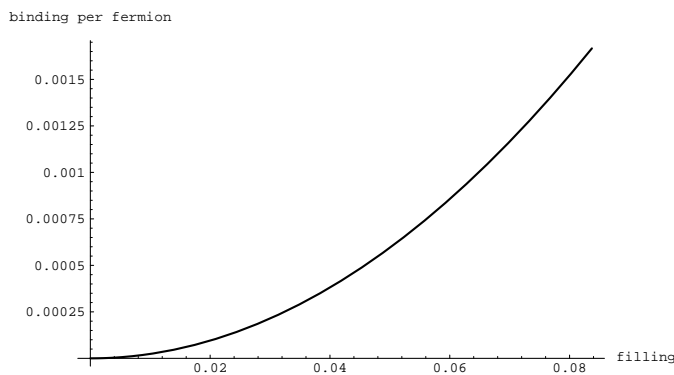


Figure 1: The binding energy per fermion  $B(\nu, \gamma)$  at  $\gamma = 0.1$

Let us consider the most stable bag, namely the bag at  $\nu = 1$ . In the small  $\gamma$  limit,

$$\frac{M(1, \gamma)}{Nm} \sim \frac{2}{\pi} - \gamma \log \frac{\pi e \gamma}{4} + \mathcal{O}(\gamma^{\frac{3}{2}}). \quad (18)$$

Thus  $M(1, \gamma)$  is non-analytic at  $\gamma = 0$ , *i.e.*, at the GN point. Note that  $M(1, 0) = \frac{2Nm}{\pi}$  is the kink-antikink threshold of the GN model [8]. It appears that the logarithmic singularity in  $M(1, \gamma)$  is associated with the enhanced  $Z_2$  symmetry at  $\gamma = 0$ . To further argue in this direction, we find that as soon as  $\nu$  decreases from 1, which puts us below the kink- antikink threshold,  $M(\nu, \gamma) = \frac{2}{\pi} \cos \frac{\pi(1-\nu)}{2} + [\log (\frac{1+\cos \frac{\pi(1-\nu)}{2}}{1-\cos \frac{\pi(1-\nu)}{2}}) - (\frac{4}{\pi(1-\nu)} - \frac{\pi(1-\nu)}{3}) \sin \frac{\pi(1-\nu)}{2}] \gamma + \mathcal{O}(\gamma^2) = \frac{2}{\pi} - 2\gamma \log \frac{\pi e(1-\nu)}{4} + \mathcal{O}((1-\nu)^2, \gamma^2, (1-\nu)^2 \gamma)$ . Thus, the logarithmic singularity  $\gamma \log \gamma$  as  $\gamma \rightarrow 0$  is replaced by a logarithmic singularity  $\gamma \log(1-\nu)$  as  $\nu \rightarrow 1$ . This means that the kink-antikink is indeed the source of this singular behavior. It would be interesting to address this issue in the framework of an appropriate effective action for bags.

We now turn to the large  $\gamma$  limit, which may be attained by making the four-fermi interactions weak. The theory should then describe quasi-free heavy fermions of mass  $m$ . We thus expect that the binding energy of bags will tend to zero as  $\gamma \rightarrow \infty$ . This is indeed the case, and we find  $\frac{M(\nu, \gamma)}{m} \sim \nu - \frac{1}{24}(\frac{\nu^3}{\gamma^2}) + \mathcal{O}(\gamma^{-3})$ . The  $\nu^3$  behavior is once again a manifestation of the mattress effect. We present the results of the numerical computation of the binding energy per fermion at maximal filling  $B(1, \gamma)$  in Fig. (2).

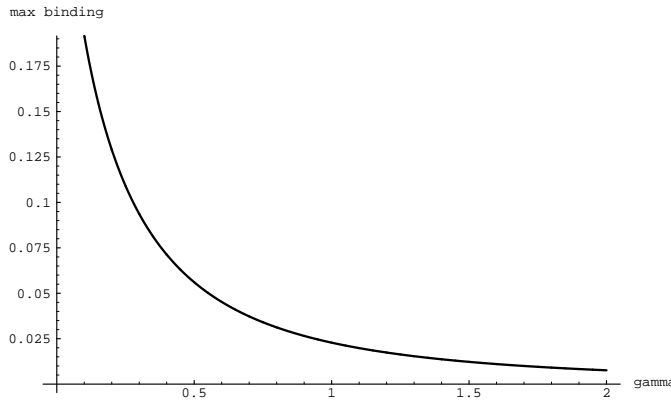


Figure 2: The binding energy per fermion  $B(1, \gamma)$  at maximal filling.



*Static bags do not exist* The extremum condition  $\delta\mathcal{E}/\delta\sigma(x) = 0$  reads

$$i \frac{\sigma(x) - M}{\lambda} = [2\sigma(x) + \partial_x] \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \langle x | \frac{1}{-\partial_x^2 + \sigma^2 - \sigma' - \omega^2} | x \rangle. \quad (19)$$

A static saddle point configuration  $\sigma(x)$  is necessarily reflectionless. Concentrating on the case of a single bound state and substituting (11) into (19) and using (13), we find

$$\frac{M(\mu)}{m\lambda(\mu)} (\sigma - m) = (2\sigma + \partial_x) (m^2 - \sigma^2 + \sigma') (-iI(\omega_b, m)/4). \quad (20)$$

Further substituting  $\sigma(x)$  as given in (9) into this equation, we verify easily that there is no combination  $\kappa(m, M, \lambda)$  for which (20) is satisfied. Thus (1) does not have a static reflectionless  $\sigma(x)$  saddle point with a single bound state. (The possibility that it has a static reflectionless saddle point with more than a single bound state seems highly unlikely, but we have not ruled it out rigorously.) We thus conclude that the MGN model does not have any static  $\sigma(x)$  configurations. This is in contrast to the GN model, which has, as we have already mentioned, static topological  $\sigma(x)$  configurations, all of them are reflectionless, with a single bound state [6, 7, 8], or more [10].

The time dependent bags in the MGN model may be seen as continuous vibrating deformations of the static non-topological bags of the GN model (at least for small values of  $\gamma$ .) Indeed, the bag configuration in the GN model with one bound state is also described by the  $\sigma$  field given in (9). Dashen et al [8] showed long ago that  $\kappa$  and  $m$  are quantized according to (16) and (17), with  $\gamma$  set to zero [6, 7]. These quantization conditions, as well as the gap equation (6), are continuous at  $\gamma = 0$ . Thus, this configuration should not change abruptly as we turn  $M$  on, but it cannot remain static. This means that as we switch on  $\gamma$ ,  $\sigma(x)$  will start vibrating around the static profile (9). The vibration amplitude and frequency of these objects must be continuous functions of  $\gamma$  that vanish as  $\gamma \rightarrow 0$ . It would be interesting to determine whether and how the lack of static  $\sigma(x)$  saddle points in the massive GN model is related to the common lore that turning  $M$  on destroys the complete integrability of the GN model.

*The fermion current operator and the bosonized theory* By using the methods of Sec. 2 of [5] we easily obtain the expectation value of the conserved fermion current  $j^\mu = \bar{\psi}\gamma^\mu\psi$  in the background of a static extremal  $\sigma(x)$  configuration trapping  $N\nu$  fermions. The spatial component is identically zero and the fermion density  $\langle\sigma|j^0|\sigma\rangle$  is

$$\rho(x) = \frac{N\nu}{4\kappa} (\sigma^2 - m^2), \quad (21)$$

which has the correct normalization  $\int_{-\infty}^{\infty} dx \rho(x) = N\nu$ , as can be seen from (9). This means that, in the bosonized theory, the flavor singlet boson  $\phi$  develops a spatially varying profile which follows the profiles of  $\sigma(x)$  according to

$$\partial_x\phi(x) = \sqrt{\frac{\pi}{8}} \frac{\nu}{\kappa} (\sigma^2 - m^2). \quad (22)$$

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